MODULI OF LANGLANDS PARAMETERS INTRODUCTION

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1. The local Langlands correspondence

1.1. **Introduction.** Let F be a finite extension of $\mathbb{Q}_p$ for some prime p and let W_F be the Weil group of F, which is a dense subgroup of the absolute Galois group of F defined by the following diagram (the topology of W_F is not the subspace topology, but rather I_F is an open subgroup of W_F giving it a locally profinite topology)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_F & \longrightarrow & W_F & \longrightarrow & \langle \mathrm{Frob}_{k_F} \rangle & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_F & \longrightarrow & \mathrm{Gal}(\overline{F}/F) & \longrightarrow & \mathrm{Gal}(\overline{k_F}/k_F) & \longrightarrow & 1. \end{array}$$

Let $\ell \neq p$ be a prime number and let $\Phi(\mathrm{GL}_{n,F})$ be the set of isomorphism classes of continuous representations

$$W_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

that are Frobenius-semisimple, i.e., that send any lift of Frob_{k_F} to a semisimple element. Recall that a representation of the topological group $\mathrm{GL}_n(F)$ on a vector space V (over any field) is called smooth if the stabiliser of every vector $v \in V$ is an open subgroup of $\mathrm{GL}_n(F)$. Let $\ell \neq p$ be a prime number and let $\Pi(\mathrm{GL}_{n,F})$ denote the set of isomorphisms classes of irreducible smooth representations of $\mathrm{GL}_n(F)$ on $\overline{\mathbb{Q}}_\ell$ vector spaces. Then the local Langlands correspondence (as proved by Harris-Taylor and Henniart) asserts that there is a bijection

$$\mathrm{LL} : \Pi(\mathrm{GL}_{n,F}) \to \Phi(\mathrm{GL}_{n,F}),$$

satisfying a long list of properties, see [2]. For example it should be given by class field theory for $n = 1$, be compatible with parabolic induction, twisting, and central characters. Moreover the bijection can be pinned down uniquely by asking that it preservers L- and ϵ-factors of pairs, see [10]. Furthermore the correspondence is 'realised' in the ℓ-adic cohomology of certain towers of rigid analytic spaces and satisfies 'local-global compatibility'.

The goal of this study group is to study (conjectural) generalisations of this result to other connected reductive groups G and to more general coefficients Λ over $\mathbb{Z}_\ell$, with our emphasis on the latter.

1.2. **More general groups.** Let G/F be a connected reductive group assumed to be split for simplicity and let $\hat{G}/\mathbb{Q}_\ell$ be the dual group of G (the one with dual root datum), for example if $G = \mathrm{GL}_n$ then $\hat{G} = \mathrm{GL}_n$, if $G = \mathrm{SO}_{2n+1}$ then $\hat{G} = \mathrm{Sp}_{2n}$ and SO_{2n} is self dual. Everything I'm about to explain should work for arbitrary quasi-split groups, although it requires replacing $\hat{G}$ with the L-group ${}^L\hat{G}$, which is what we will be doing in the paper. Doing things for non quasi-split groups is harder, even conjecturally, see [11].

We define $\Pi(G)$ as before as the set of isomorphism classes of irreducible smooth representations of the topological group $G(F)$ as before. For $\Phi(G)$ we take the set of $\hat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of continuous group homomorphisms

$$\phi : W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$$

such that Frobenius elements in W_F map to semisimple elements of the target.

1.2.1. Then conjecturally there is a surjective map (surjectivity should only hold in the quasi-split case)

$$\mathrm{LL} : \Pi(G) \to \Phi(G)$$

with finite fibers $\Pi_\phi = \mathrm{LL}^{-1}(\phi)$ called L-packets that can be described explicitly I don't know if I have to restrict to the tempered case for either of these statements. For example if $G = \mathrm{GL}_{n,F}$ then these L-packets are singletons and the map is a bijection as above. In the general quasi-split case these L-packets should be in bijection with the representations of a finite group S_ϕ attached to ϕ.

There is a long list of desiderata for the map LL, see [11], however it is not clear that these determine LL uniquely if it exists.

1.2.2. One way to 'rigidity' our conjectural local Langlands correspondence is to no longer phrase things in terms of bijections of sets, but rather as an equivalence of (stable ∞)-categories, for example Conjecture X.3.5 of [6] or Conjecture 4.6.4 of [14]. Similarly work of Emerton-Helm suggests that the local Langlands correspondence should interpolate 'in families', i.e., with more general coefficients than $\overline{\mathbb{Q}}_\ell$ such as $\overline{\mathbb{Z}}_\ell$ or more general $\mathbb{Z}_\ell$-algebras [or even $\mathbb{Z}[1/p]$-algebras].

1.3. **The Bernstein centre.** Let

$$\mathrm{Rep}^{\mathrm{sm}}_{\overline{\mathbb{Z}}_\ell}(G(F))$$

be the category of smooth representations of $G(F)$ on $\overline{\mathbb{Z}}_\ell$-modules. This is a $\overline{\mathbb{Z}}_\ell$-linear abelian category, and therefore the set of endomorphisms of the identity functor

$$\mathcal{Z}(G(F), \overline{\mathbb{Z}}_\ell) := \mathrm{End}(\mathrm{Id}_{\mathrm{Rep}^{\mathrm{sm}}_{\overline{\mathbb{Z}}_\ell}(G(F))})$$

has the structure of a commutative $\overline{\mathbb{Z}}_\ell$-algebra, called the (integral ℓ-adic) *Bernstein center* of $G(F)$. There is also a version with $\overline{\mathbb{Q}_\ell}$-coefficients, which is well understood thanks to work of Bernstein [1]. There is also work of Vigneras for $\overline{\mathbb{F}}_\ell$-coefficients [13], although there the theory is less complete when $G \neq \mathrm{GL}_{n,F}$. Again in the case that $G = \mathrm{GL}_{n,F}$ there is extensive work of David Helm [7, 8] on the structure of $\mathcal{Z}(G(F), \overline{\mathbb{Z}}_\ell)$.

Now let $k = \overline{\mathbb{Z}}_\ell$ or $\overline{\mathbb{Q}}_\ell$ and let π be an object in $\mathrm{Rep}^{\mathrm{sm}}_k(G(F))$ satisfying $\mathrm{End}(\pi) = k$ (lets call such π Schur-irreducible). Then π determines a ring homomorphism

$$\mathcal{Z}(G(F), k) \to k.$$

1.3.1. *The spectral Bernstein centre.* The goal of this study group is to define- and study the spectral Bernstein centre

$$\mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Z}}_\ell).$$

Let $\hat{G}/\overline{\mathbb{Z}}_\ell$ be the dual group of G. The spectral Bernstein centre will be the ring of global functions on the stack

$$\left[Z^1(W_F, \hat{G})/\hat{G} \right],$$

where $Z^1(W_F, \hat{G})$ represents the functor whose R-points for a $\overline{\mathbb{Z}}_\ell$-algebra R is the set of continuous homomorphisms (which we will call $\hat{G}$-valued L-parameters)

$$W_F \to \hat{G}(R),$$

where $\hat{G}(R)$ is topologised in a certain way. We will show that $Z^1(W_F, \hat{G})$ is flat $\overline{\mathbb{Z}}_\ell$ of relative dimension equal to $\mathrm{Dim}\, \hat{G}$ and is a complete intersection. We will show that ring homomorphisms $\mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Z}}_\ell) \to k$ correspond to semisimple continuous homomorphisms $W_F \to \hat{G}(k)$ for $k = \overline{\mathbb{Z}}_\ell$ or $\overline{\mathbb{Q}}_\ell$ as above.

We will also study the connected component of the moduli space of L-parameters, which will be done by restricting our L-parameters to wild-inertia (over $\mathbb{Z}[1/p]$), over restricting to prime-to-λ inertia (over $\mathbb{Z}_\lambda$) which leads to a sort of theory of inertial types.

1.3.2. *A semisimple local Langlands correspondence.* One of the main results of Fargues-Scholze [6] is that there is a ring homomorphism (with a lot of good properties, like compatibility with central extensions, twisting and parabolic induction)

$$\Psi_G : \mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Z}}_\ell) \to \mathcal{Z}(G(F), \overline{\mathbb{Z}}_\ell).$$

In particular for every Schur-irreducible smooth $G(F)$-representation π with coefficients in k we get a semisimple continuous homomorphism

$$\varphi_{\mathrm{FS},\pi} : W_F \to \hat{G}(k).$$

The association $\pi \mapsto \varphi_{\mathrm{FS},\pi}$ satisfies a list of properties (compatibility with parabolic induction etc) and is also compatible with the cohomology of Shimura varieties by work of Koshikawa [12]. When $G = \mathrm{GL}_n$ and $\pi/\overline{\mathbb{Q}}_\ell$ is supercuspidal, this recovers the usual local Langlands parameter of π and in fact the morphism Ψ_{GL_n} is an isomorphism. This recovers work of Helm-Moss [9] and Emerton-Helm [5] on the local Langlands correspondence in families.

1.4. Actually the moduli stack of $\hat{G}$-valued Langlands parameters can even be defined over $\mathbb{Z}[1/p]$ by choosing a discretisation of the tame inertia group as done in [3]. It should be said that Fargues-Scholze do much more than constructing the morphisms Ψ_G, and in fact they give a very interesting conjectural categorical version of the local Langlands correspondence. However understanding this conjecture would require studying their work in detail, and the existence of Ψ_G is something that can be black-boxed very easily.

Very recently the existence of Ψ_G together with the properties of $\mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Z}}_\ell)$ has lead to new finiteness results for smooth representations of $G(F)$ over modules over $Z[1/p]$-algebras, see [4]; for example this shows that parabolic induction preserves finitely generated objects. This paper will be discussed by Lie in the final talk of the learning seminar.

2. Weil-Deligne representations

A classical observation is that the notion of smooth representation of $\mathrm{GL}_n(F)$ does not depend on the ℓ-adic topology on the coefficient field, while the notion of an n-dimension representations of W_F over $\overline{\mathbb{Q}}_\ell$ does (a priori). Weil-Deligne representations were invented to fix this and so we will spend some time discussing them, a good source is `https://math.commelin.net/files/wdreps.pdf`. For an element $w \in W_F$ we will write $||w|| \in q^{\mathbb{Z}}$ for the image of $w \in W_F/I_F = \mathbb{Z}$ and where $q = |k_F|$ is the cardinality of the residue field of F.

2.1. **Monodromy is quasi-unipotent.** We start with a motivating theorem:

Theorem 2.1.1. *Let K be a finite extension of $\mathbb{Q}_\ell$ and let (ρ, V) be a finite-dimensional representation of W_F over K. Then there exists an open subgroup $H \subset I_F$ such that $\rho(x)$ is unipotent for all $x \in H$.*

Proof. The maximal unramified extension $F^{\mathrm{ur}} \subset \overline{F}$ has a unique (up to isomorphism) extension of degree n for n coprime to p. This extension is Galois and it is given by the splitting field of $X^n - \varpi_F$, where ϖ_F is a uniformiser of F. This translates into the statement that the quotient I_F/P_F of I_F by the wild inertia subgroup P_F is isomorphic to

$$\prod_{\lambda \neq p} \mathbb{Z}_\lambda(1),$$

where the twist (1) denotes the fact that the action of $W_F/I_F = \langle \mathrm{Frob}_{k_F} \rangle$ given by lifting an element to W_F and then acting by conjugation is given by the ℓ-adic cyclotomic character.

By the continuity of ρ, we can find an open subgroup $H \subset W_F$ such that $\rho(H)$ lands in $1 + \ell^2 M_{n\times n}(\mathcal{O}_K) \subset \mathrm{GL}_n(K)$. This means that the image of $I_F \cap H$ is a pro-ℓ group and therefore factors over t_ℓ, where t_ℓ is the map

$$I_F \to \prod_{\lambda \neq p} \mathbb{Z}_\lambda(1) \to \mathbb{Z}_\ell(1).$$

It is not so hard to see that for each $x \in (H \cap I_F)$ (using that $\rho(x) - 1$ is divisible by ℓ^2)

$$\log(\rho(x)) := \sum_{j=0}^{\infty} (-1)^{j-1} \frac{(\rho(x) - 1)^j}{j}$$

converges. Because the restriction of ρ to $I_F \cap H$ factors through t_ℓ we find that for any $w \in W_F$ we get

$$\log(\rho(wxw^{-1})) = \log(\rho(x^{||w||})) = ||w|| \log \rho(x).$$

On the other hand since logarithms of matrices commutes with conjugation, we see that the matrices $\log \rho(x)$ and $||w|| \log \rho(x)$ are conjugate. In particular if we take w to be a lift of Frobenius then $||w|| = q$ and we find that the $n-k$-th coefficient a_{n-k} of the characteristic polynomial of $\log \rho(x)$ satisfies

$$q^k a_{n-k} = a_{n-k},$$

which implies that $a_{n-k} = 0$ for $k > 0$ and thus $\log \rho(x)$ is nilpotent. $\square$

Corollary 2.1.2. *Let K and (ρ, V) be as in the statement of the theorem. Then there exists a unique nilpotent operator $N \in \mathrm{End}_K(V)(-1)$ such that*

$$\rho(x) = \mathrm{Exp}(t_\ell(x) N),$$

for all x in some open subgroup $H \subset I_F$.

Proof. The uniqueness is clear because it is asserting that

$$\log \rho(x) = t_\ell(x) \cdot N$$

and so N is determined by $\log \rho(x)$ for a single x.

Existence is clear if there is an open subgroup $H \subset W_F$ such that the restriction of ρ to $H \cap I_F$ is trivial (take $N = 0$). If such an H does not exist then we can find $y \in H \cap I_F$ for a fixed H such that $t_\ell(y) \neq 0$ and we can define

$$N = \log \rho(y) \cdot t_\ell(y)^{-1}.$$

Now shrink H such that the restriction σ of ρ to $I_F \cap H$ factors through $t_\ell(x)$. The representation of $Z_\ell(1)$ given by

$$x \mapsto \exp(Nx)$$

agrees with the map induced by σ on the subgroup $t_\ell(y)\mathbb{Z}_\ell(1)$, which is of finite index in $\mathbb{Z}_\ell(1)$ and so we win after potentially shrinking H again. $\square$

2.2. **The definition.** We are now ready to define Weil-Deligne representations

Definition 2.2.1. *A Weil-Deligne representation over a field (of characteristic zero) is a triple (ρ, V, N) where (ρ, V) is a finite dimensional representation of W_F whose restriction to I_F factors through a finite quotiemt and $N \in \mathrm{End}(V)$ is a nilpotent endomorphism such that for any $w \in W$ we have*

$$\rho(w) N \rho(w)^{-1} = ||w|| \cdot N.$$

A morphism of Weil-Deligne representations is a morphism f of representations such that $f \circ N = N' \circ f$. We say that a Weil-Deligne representation is Frobenius-semisimple if the representation ρ is semisimple.

Given a Weil-Deligne representation (ρ, V, N) over a finite extension K of $\mathbb{Q}_\ell$ we can define a continuous representation ρ' of W_F by

$$\rho'(\Phi^a \cdot x) = \rho(x) \cdot \exp(t_\ell(x) N)$$

with $x \in I_F$ and Φ a lift of Frobenius and in fact this induces an equivalence of categories between continuous finite-dimensional representations of W_F on K-vector spaces an Weil-Deligne representations (V, ρ, N) of W_F over K.

2.3. **More general reductive groups.** Let G/F be a split connected reductive group and let K be a finite extension over $\mathbb{Q}_\ell$ with $\ell \neq p$.

Definition 2.3.1. *A Weil-Deligne representation with values in $\hat{G}$ is a pair (r, N) where $r : W_F \to \hat{G}(\mathbb{Q}_\ell)$ is a group homomorphism whose restriction to I_F factors through a finite quotient, and $N \in \hat{\mathfrak{g}}_{\mathbb{Q}_\ell}$ is an element such that*

$$\mathrm{Ad}\, r(g) N = ||g|| \cdot N$$

for all $g \in W_F$.

Note that the element N is automatically nilpotent, for example because its image in gl_n is nilpotent for every representation $G \to \mathrm{GL}_n$. Note moreover that this definition makes sense with coefficients in an arbitrary $\mathbb{Q}_\ell$-algebra.

More concretely if we fix a finite extension L/F such that the restriction of r to $I_L \subset I_F$ is trivial. Then a Weil-Deligne representation (whose restriction to I_L is trivial is simply a triple (Φ, N, τ) where $\Phi \in \hat{G}(\overline{\mathbb{Q}}_\ell)$, where N is as before and where $\tau : I_{L/K} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ is a group homomorphism such that

$$\begin{aligned}\mathrm{Ad}(\tau(g))N &= N\\ \mathrm{Ad}(\Phi)N &= q^{-1}\cdot N\\ \mathrm{Ad}(\Phi)\tau(\gamma) &= \tau(\phi\gamma\phi^{-1}),\end{aligned}$$

where ϕ is a fixed lift of Frobenius in W_F. The moduli space of Weil-Deligne representations will then live over

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